Ostrogradski formalism for higher-derivative scalar field theories



1998 J. Phys. A: Math. Gen. 31 6949

(http://iopscience.iop.org/0305-4470/31/33/006)

J. Phys. A: Math. Gen. **31** (1998) 6949–6964. Printed in the UK PII: S0305-4470(98)94586-3

# Ostrogradski formalism for higher-derivative scalar field theories

F J de Urries†‡ and J Julve‡

† Departamento de Física, Universidad de Alcalá de Henares, 28871 Alcalá de Henares, Madrid, Spain

‡ IMAFF, Consejo Superior de Investigaciones Científicas, Serrano 113 bis, Madrid 28006, Spain

Received 26 May 1998

**Abstract.** We carry out the extension of the Ostrogradski method to relativistic field theories. Higher-derivative Lagrangians reduce to second differential order with one explicit independent field for each degree of freedom. We consider a higher-derivative relativistic theory of a scalar field and validate a powerful order-reducing covariant procedure by a rigorous phase-space analysis. The physical and ghost fields appear explicitly. Our results strongly support the formal covariant methods used in higher-derivative gravity.

## 1. Introduction

Theories with higher-order Lagrangians have an old tradition in physics, and Podolski's generalized electrodynamics [1] (later visited as a useful testbed [2]), effective gravity and tachyons [3] are examples. Interest in higher-order mechanical systems is still alive today [4].

Theories of gravity with terms of any order in curvatures arise as part of the low-energy effective theories of strings [5] and from the dynamics of quantum fields in a curved spacetime background [6]. Theories of second-order (four-derivative theories in the following) have been studied more closely in the literature because they are renormalizable [7] in four dimensions. Phenomenological applications arose that spurred further theoretical interest, as illustrated by the most comprehensive introductory study available [8]. In fact, they greatly affect the effective potential and phase transitions of scalar fields in curved spacetime, with a wealth of astrophysical and cosmological properties [9]. A procedure based on the Legendre transformation was devised [10] to recast them as an equivalent theory of second differential order. A suitable diagonalization of the resulting theory was found later [11] that yields the explicit independent fields for the degrees of freedom (DOF) involved, usually including Weyl ghosts.

In [12] the simplest example of this procedure was given using a model of one scalar field with a massless and a massive DOF. In an appendix, Barth and Christensen [13] gave the splitting of the higher-derivative (HD) propagator into quadratic ones for the fourth, sixth and eighth differential-order scalar theories. A scalar six-derivative theory has been considered in [14] as a regularization of the Higgs model, yielding a finite theory.

Classical treatises [15] study the Lagrangian and Hamiltonian theories of systems with a finite number of DOF and higher time derivatives of the generalized coordinates. Later work has considered the variational problem of those theories with the tools of the Cartan form, k-jets, symplectic geometry and Legendre mappings [16].

However, the particular case of relativistic covariant field theories has complications of its own which are not covered by those general treatments. Our presentation highlights the Lorentz covariance and the particle aspect of the theory, with emphasis in the structure of the propagators and the coupling to other matter sources. We address this issue by using a simplified model with scalar fields as in [12, 13], and our extrapolation of the canonical analysis to these continous systems validates the formal procedures introduced there. The analysis presented here mostly focuses on the free part of the Lagrangian, and self-interactions and interactions with other fields are embodied in a source term.

In section 2 we briefly review the Ostrogradski method and outline our extension to the field theories. In section 3 we study the case of the four-derivative theory for arbitrary non-degenerate masses, which exemplifies the use of the Helmholtz Lagrangian and the crucial diagonalization of the fields. The eight-derivative case and higher $4N$-derivative cases are considered in section 4. For even $N$ the $2N$-derivative cases present some peculiarities that deserve the separate discussion of section 5. Our results are summarized in the conclusion.

As a general feature, our procedure involves vectors with pure real and imaginary components as well as symmetric matrices with equally assorted elements. Diagonalizing symmetric matrices of this kind is a non-standard task which is detailed in the appendix.

## 2. Ostrogradski's method

We consider a HD Lagrangian theory for a system described by configuration variables $q(t)$. By dropping total derivatives, it can always be brought to a standard form

$$L[q, \dot{q}, \ddot{q}, \ldots, \overset{(m)}{q}] \tag{2.1}$$

depending on time derivatives of the lowest possible order. The variational principle then yields equations of motion which are of differential order $2m$ at most:

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}} + \cdots + (-1)^m \frac{\mathrm{d}^m}{\mathrm{d}t^m}\frac{\partial L}{\partial \overset{(m)}{q}} = 0. \tag{2.2}$$

Hamilton's equations are obtained by defining $m$ generalized momenta

$$\begin{aligned} p_m &\equiv \frac{\partial L}{\partial \overset{(m)}{q}} \\ p_i &\equiv \frac{\partial L}{\partial \overset{(i)}{q}} - \frac{\mathrm{d}}{\mathrm{d}t} p_{i+1} \qquad (i = 1, \ldots, m-1) \end{aligned} \tag{2.3}$$

and $m$ independent variables

$$\begin{aligned} q_1 &\equiv q \\ q_i &\equiv \overset{(i-1)}{q} \qquad (i = 2, \ldots, m). \end{aligned} \tag{2.4}$$

Then the Lagrangian may be considered to depend on the coordinates $q_i$ and only on the first time derivative $\dot{q}_m = \overset{(m)}{q}$. A Hamiltonian on the phase space $[q_i, p_i]$ may then be found by working $\dot{q}_m$ out of the first equation (2.3) as a function

$$\dot{\boldsymbol{q}}_{\boldsymbol{m}}[q_1, \ldots, q_m; p_m] \tag{2.5}$$

the remaining velocities $\dot{q}_i$ $(i = 1, \ldots, m-1)$ already being expressed in terms of coordinates, because of (2.4), as

$$\dot{\boldsymbol{q}}_{\boldsymbol{i}} = q_{i+1}. \tag{2.6}$$

Thus

$$H[q_i, p_i] = \sum_{i=1}^{m} p_i \dot{\boldsymbol{q}}_{\boldsymbol{i}} - L = \sum_{i=1}^{m-1} p_i q_{i+1} + p_m \dot{\boldsymbol{q}}_{\boldsymbol{m}} - L[q_1, \ldots, q_m; \dot{\boldsymbol{q}}_{\boldsymbol{m}}]. \tag{2.7}$$

Therefore

$$\delta H = \sum_{i=1}^{m-1} (p_i \delta q_{i+1} + q_{i+1} \delta p_i) + p_m \delta \dot{\boldsymbol{q}}_{\boldsymbol{m}} + \dot{\boldsymbol{q}}_{\boldsymbol{m}} \delta p_m - \sum_{i=1}^{m} \frac{\partial L}{\partial q_i} \delta q_i - \frac{\partial L}{\partial \dot{\boldsymbol{q}}_{\boldsymbol{m}}} \delta \dot{\boldsymbol{q}}_{\boldsymbol{m}} \tag{2.8}$$

but (2.3) can be written as

$$\begin{aligned} &\frac{\partial L}{\partial \dot{\boldsymbol{q}}_{\boldsymbol{m}}} = p_m \\ &\frac{\partial L}{\partial q_i} = \dot{p}_i + p_{i-1} \qquad (i = 2, \ldots, m) \end{aligned} \tag{2.9}$$

and (2.2), because of (2.3), gives

$$\frac{\partial L}{\partial q_1} = \frac{\partial L}{\partial q} = \dot{p}_1 \tag{2.10}$$

so we obtain

$$\delta H = \sum_{i=1}^{m} (-\dot{p}_i \delta q_i + \dot{q}_i \delta p_i) \tag{2.11}$$

and the canonical equations of motion turn out to be

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}. \tag{2.12}$$

Summarizing we may say that a theory with one configuration coordinate $q$ obeying equations of motion of $2m$ differential order (stemming from a Lagrangian with quadratic terms in $\overset{(m)}{q}$ as its highest-derivative dependence) can be cast as a set of first-order canonical equations for $2m$ phase-space variables $[q_i, p_i]$.

As is well known, once the differential order has been reduced by the Hamiltonian formalism, one may prefer to obtain the same canonical equations of motion from a variational principle. Then the canonical equations (2.12) are the Euler equations of the so-called Helmholtz Lagrangian

$$L_H[q_i, \dot{q}_i, p_i] = \sum_{i=1}^{m} p_i \dot{q}_i - H[q_i, p_i] \tag{2.13}$$

which depends on the $2m$ coordinates $q_i$ and $p_i$, and only on the velocities $\dot{q}_i$. This alternative setting will be adopted later on.

As far as finite-dimensional mechanical systems are concerned, only time derivatives are involved. The generalized momenta above have a mechanical meaning and the resulting Hamiltonian is the energy of the system up to problems of positiveness linked to the occurrence of ghost states.

### 2.1. Extension to field theories

Continuous systems with field coordinates $\phi(t, \boldsymbol{x})$ usually involve space derivatives as well, chiefly if relativistic covariance is assumed. We now generalize the previous formalism to this case. A HD field Lagrangian density will have the general dependence

$$\mathcal{L}[\phi, \phi_\mu, \ldots, \phi_{\mu_1\ldots\mu_m}] \tag{2.14}$$

where $\phi_{\mu_1\ldots\mu_i} \equiv \partial_{\mu_1}\ldots\partial_{\mu_i}\phi$, with corresponding equations of motion

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\phi_\mu} + \cdots + (-1)^m\partial_{\mu_1}\ldots\partial_{\mu_m}\frac{\partial\mathcal{L}}{\partial\phi_{\mu_1\ldots\mu_m}} = 0. \tag{2.15}$$

The generalized momenta now are

$$\begin{aligned} \pi^{\mu_1\ldots\mu_m} &\equiv \frac{\partial\mathcal{L}}{\partial\phi_{\mu_1\ldots\mu_m}} \\ \pi^{\mu_1\ldots\mu_i} &\equiv \frac{\partial\mathcal{L}}{\partial\phi_{\mu_1\ldots\mu_i}} - \partial_{\mu_{i+1}}\pi^{\mu_1\ldots\mu_i\mu_{i+1}} \qquad (i = 1, \ldots, m-1). \end{aligned} \tag{2.16}$$

Though they do not have a direct mechanical meaning of impulses they still are suitable to perform a Legendre transformation upon.

Assuming also that the highest derivative can be worked out of the first equation of (2.16) as a function $\bar{\phi}_{\mu_1\ldots\mu_m}[\phi, \phi_\mu, \ldots, \phi_{\mu_1\ldots\mu_{m-1}}; \pi^{\mu_1\ldots\mu_m}]$, the 'Hamiltonian' density is now

$$\begin{aligned} \mathcal{H}[\phi, \phi_\mu, \ldots, \phi_{\mu_1\ldots\mu_{m-1}}; \pi^\mu, \ldots, \pi^{\mu_1\ldots\mu_m}] &= \pi^\mu\phi_\mu + \cdots + \pi^{\mu_1\ldots\mu_{m-1}}\phi_{\mu_1\ldots\mu_{m-1}} \\ &\quad + \pi^{\mu_1\ldots\mu_m}\bar{\phi}_{\mu_1\ldots\mu_m} - \mathcal{L}[\phi, \phi_\mu, \ldots, \bar{\phi}_{\mu_1\ldots\mu_m}]. \end{aligned} \tag{2.17}$$

Then the canonical equations are

$$\begin{aligned} &\partial_\mu\phi = \frac{\partial\mathcal{H}}{\partial\pi^\mu}, \partial_\mu\phi_\nu = \frac{\partial\mathcal{H}}{\partial\pi^{\mu\nu}}, \ldots, \partial_\mu\phi_{\mu_1\ldots\mu_{m-1}} = \frac{\partial\mathcal{H}}{\partial\pi^{\mu\mu_1\ldots\mu_{m-1}}}, \\ &\partial_\mu\pi^\mu = -\frac{\partial\mathcal{H}}{\partial\phi}, \partial_\nu\pi^{\mu\nu} = -\frac{\partial\mathcal{H}}{\partial\phi_\mu}, \ldots, \partial_\sigma\pi^{\mu_1\ldots\mu_{m-1}\sigma} = -\frac{\partial\mathcal{H}}{\partial\phi_{\mu_1\ldots\mu_{m-1}}}. \end{aligned} \tag{2.18}$$

This general setting may be hardly applicable to systems of practical interest (generally involving internal symmetries and/or fields belonging to less trivial Lorentz representations) if suitable strategies are not adopted to refine the method. One crucial observation is that the momenta may be defined in more useful and general ways than the plain one introduced in (2.16): instead of differentiating with respect to the simple field derivatives $\phi_{\mu_1\ldots\mu_i}$ one may consider combinations of field derivatives of different orders belonging to the same Lorentz and internal group representations. For instance, in HD gravity [10], the Ricci tensor is the most suited combination of second derivatives of the metric tensor field. The only condition is that the Lagrangian be regular in the highest 'velocity' so defined. This will be made clear in the following.

In fact this general Ostrogradski treatment can be significantly simplified for the Lorentz invariant theory of a scalar field, which is the example we will consider in this paper. In this case, dropping total derivatives, the general form (2.14) can be expressed in a more convenient way that singles out the free quadratic part, namely

$$\mathcal{L} = -\frac{c}{2}\phi[\![1]\!][\![2]\!]\ldots[\![N]\!]\phi - j\phi \tag{2.19}$$

where $[\![i]\!] \equiv (\Box + m_i^2)$, our Minkowski signature is $(+, -, -, -)$ so that $\Box \equiv \partial_t^2 - \triangle$, and $c$ is a dimensional constant. The masses are ordered such that $m_i > m_j$ when $i < j$ so that the objects $\langle ij\rangle \equiv (m_i^2 - m_j^2)$ are always positive when $i < j$.

It turns out to be very advantageous to consider only Lorentz invariant combinations of derivatives of the type $\Box^n\phi$ and of the $\phi$ field itself with suitable dimensional coefficients. Further, it is even more useful to consider expressions of the form $[\![i]\!]^n\phi$.

Thus, arbitrarily focusing ourselves on $i = 1$ without loss of generality, equation (2.19) may be recast as

$$\mathcal{L} = \tfrac{1}{2}\sum_{n=1}^{N} c_n\phi[\![1]\!]^n\phi - j\phi \tag{2.20}$$

where the $c_n$ are redefined constants.

Calling $m = \frac{N}{2}$ for even $N$, and $m = \frac{N+1}{2}$ for odd $N$, the motion equation now reads

$$\sum_{n=1}^{m}[\![1]\!]^n\frac{\partial\mathcal{L}}{\partial([\![1]\!]^n\phi)} = \sum_{n=1}^{N} c_n[\![1]\!]^n\phi - j. \tag{2.21}$$

The Legendre transform can now be performed upon the simpler set of *generalized momenta*

$$\begin{aligned}
\pi_m &= \frac{\partial\mathcal{L}}{\partial([\![1]\!]^m\phi)}\\
\pi_{m-1} &= \frac{\partial\mathcal{L}}{\partial([\![1]\!]^{m-1}\phi)} + [\![1]\!]\pi_m\\
\cdots &\quad \cdots\\
\pi_s &= \frac{\partial\mathcal{L}}{\partial([\![1]\!]^s\phi)} + [\![1]\!]\pi_{s+1} \qquad (s = 1,\ldots,m-2).
\end{aligned} \tag{2.22}$$

The Hamiltonian will depend on the new phase-space coordinates $H[\phi_1,\ldots,\phi_m;\pi_1,\ldots,\pi_m]$, where $\phi_i \equiv [\![1]\!]^{i-1}\phi$. To this end $[\![1]\!]^m\phi$ has been worked out of the first (2.22) for even $N$, or of the second (2.22) for odd $N$, in terms of these coordinates.

The dynamics of the system is given by the $2m$ equations of second order

$$\begin{aligned}
[\![1]\!]\phi_i &= \frac{\partial H}{\partial\pi_i}\\
[\![1]\!]\pi_i &= \frac{\partial H}{\partial\phi_i} \qquad (i = 1,\ldots,m).
\end{aligned} \tag{2.23}$$

Notice that, in comparison with (2.12), (2.16) and (2.18), no negative sign occurs in both (2.22) and (2.23), because each step now involves two derivative orders.

As a final comment, the treatment followed above keeps Lorentz invariance explicitly, and this will turn out to be advantageous later on. The price has been that neither do the $\pi$'s have the meaning of mechanical momenta nor does $H$ depend on the energy of the system. However, they are adequate for providing a set of 'canonical' equations that correctly describe the evolution of the system. Moreover, these equations are Lorentz invariant and of second differential order, which will lend itself to an almost direct particle interpretation.

One may, however, choose to work with the genuine Hamiltonian and mechanical momenta obtained when the Legendre transformation built-in in the Ostrogradski method involves only the true 'velocities' $\partial_t^n\phi$. The price now is losing the explicit Lorentz invariance and facing more cumbersome calculations, as we will see by an example in the second part of the next section.

## 3. $N = 2$ theories

These theories allow a particularly simple treatment that will be illustrated in the examples $N = 2$ and $N = 4$. The equations (2.23) for $N = 2$ will now be obtained from a Helmholtz-like Lagrangian of second differential order, which is closer to a direct particle interpretation.

Consider the $N = 2$ Lagrangian

$$\mathcal{L}^4 = -\frac{1}{2}\frac{1}{M}\phi[\![1]\!][\![2]\!]\phi - j\phi \tag{3.1}$$

with non-degenerate masses $m_1 > m_2$. Taking the dimensional constant $M = (m_1^2 - m_2^2) \equiv \langle 12\rangle > 0$, equation (3.1) yields the propagator

$$-\frac{\langle 12\rangle}{[\![1]\!][\![2]\!]} = \frac{1}{[\![1]\!]} - \frac{1}{[\![2]\!]}. \tag{3.2}$$

We thus see that the pole at $m_2$ then corresponds to a physical particle and the one at $m_1$ to a negative norm 'poltergeist'. The second-order Lagrangian we are seeking should describe two fields with precisely the particle propagators occurring in the r.h.s. of (3.2).

The Lagrangian (3.1) can be brought to the form (2.20), namely

$$\begin{aligned}\mathcal{L}^4[\phi, [\![1]\!]\phi] &= -\frac{1}{2}\frac{1}{\langle 12\rangle}[\phi[\![1]\!]^2\phi - \langle 12\rangle\phi[\![1]\!]\phi] - j\phi \\ &= -\frac{1}{2}\frac{1}{\langle 12\rangle}[([\![1]\!]\phi)^2 - \langle 12\rangle\phi([\![1]\!]\phi)] - j\phi\end{aligned} \tag{3.3}$$

where the relationship $[\![2]\!] = [\![1]\!] - \langle 12\rangle$ has been used.

We define one momentum

$$\pi = \frac{\partial\mathcal{L}}{\partial([\![1]\!]\phi)} \tag{3.4}$$

from which $[\![1]\!]\phi$ is readily worked out, obtaining

$$\mathcal{H}^4[\phi, \pi] = -\tfrac{1}{2}\langle 12\rangle(-\pi + \tfrac{1}{2}\phi)^2 + j\phi \tag{3.5}$$

and the Helmholtz-like Lagrangian is

$$\mathcal{L}_H^4[\phi, [\![1]\!]\phi, \pi] = \pi[\![1]\!]\phi - \mathcal{H}[\phi, \pi]. \tag{3.6}$$

It contains mixed terms $\pi\phi$ that obscure the particle contents. The diagonalization is achieved by new fields $\phi_1$, $\phi_2$

$$\begin{aligned}\phi &= \phi_1 + \phi_2 \\ \pi &= \tfrac{1}{2}(\phi_1 - \phi_2)\end{aligned} \tag{3.7}$$

to yield

$$\mathcal{L}^2 = \tfrac{1}{2}\phi_1[\![1]\!]\phi_1 - \tfrac{1}{2}\phi_2[\![2]\!]\phi_2 - j(\phi_1 + \phi_2) \tag{3.8}$$

where the particle propagators in the r.h.s. of (3.2) are apparent. This result is physically meaningful: where we had a single field $\phi$, coupled to a source $j$, propagating with the quartic propagator in the l.h.s. of (3.2) as implied by the HD Lagrangian (3.1), we now have two fields $\phi_1$, $\phi_2$ describing particles with quadratic propagators, and the source couples to the sum $\phi_1 + \phi_2$.

A deeper insight of the phase-space structure of the theory can be achieved by the plain use of the Ostrogradski method, eventually confirming the final form (3.8). In order to explicitly show the velocities, we write (3.1) in the form of the Lagrangian density

$$\mathcal{L}^4 = -\frac{1}{2}\frac{1}{\langle 12\rangle}\{(\partial_t^2\phi)^2 - (\partial_t\phi)S(\partial_t\phi) + \phi P\phi\} - j\phi \tag{3.9}$$

where $S \equiv M_1^2 + M_2^2,\ P \equiv M_1^2 M_2^2$ and $M_i^2 \equiv m_i^2 - \triangle$ are operators containing the space derivatives.

The Ostrogradski formalism yields the Hamiltonian density

$$\mathcal{H}^4[\phi, \dot\phi; \pi_1, \pi_2] = -\frac{1}{2}\langle 12\rangle\pi_2^2 + \pi_1\dot\phi - \frac{1}{2}\frac{1}{\langle 12\rangle}\dot\phi S\dot\phi + \frac{1}{2}\frac{1}{\langle 12\rangle}\phi P\phi + j\phi \tag{3.10}$$

that depends on the phase-space coordinates $\phi, \dot\phi, \pi_1, \pi_2$ and on their space derivatives. The highest-order 'velocity' $\partial_t^2\phi$ has been worked out of the momenta

$$\begin{aligned}\pi_2 &\equiv \frac{\partial\mathcal{L}^4}{\partial(\partial_t^2\phi)} = -\frac{1}{\langle 12\rangle}\partial_t^2\phi\\ \pi_1 &\equiv \frac{\partial\mathcal{L}^4}{\partial(\partial_t\phi)} - \partial_t\pi_2.\end{aligned} \tag{3.11}$$

The canonical equations may be derived from the Helmholtz Lagrangian

$$\begin{aligned}\mathcal{L}_H^4[\phi, \dot\phi; \pi_1\pi_2; \partial_t\phi, \partial_t\dot\phi] &= \pi_2\partial_t\dot\phi + \pi_1\partial_t\phi + \frac{1}{2}\langle 12\rangle\pi_2^2 - \pi_1\dot\phi\\ &\quad + \frac{1}{2}\frac{1}{\langle 12\rangle}\dot\phi S\dot\phi - \frac{1}{2}\frac{1}{\langle 12\rangle}\phi P\phi - j\phi.\end{aligned} \tag{3.12}$$

This is a Lagrangian density of first order in time derivatives, and we express it in matrix form for later convenience:

$$\mathcal{L}_H^4 = \tfrac{1}{2}\Phi^T\mu\Sigma\partial_t\Phi + \tfrac{1}{2}\Phi^T\mathcal{M}_4\Phi - J^T\Phi \tag{3.13}$$

where $\mu$ is an arbitrary mass parameter and

$$\begin{aligned}\Phi &\equiv \begin{pmatrix}\pi_2\\ \mu^{-1}\dot\phi\\ \mu^{-1}\pi_1\\ \phi\end{pmatrix} \qquad \Sigma \equiv \begin{pmatrix}0&1&0&0\\ -1&0&0&0\\ 0&0&0&1\\ 0&0&-1&0\end{pmatrix}\\ \mathcal{M}_4 &\equiv \begin{pmatrix}\langle 12\rangle&0&0&0\\ 0&\frac{\mu^2 S}{\langle 12\rangle}&-\mu^2&0\\ 0&-\mu^2&0&0\\ 0&0&0&-\frac{P}{\langle 12\rangle}\end{pmatrix} \qquad J \equiv \begin{pmatrix}0\\0\\0\\j\end{pmatrix}\end{aligned} \tag{3.14}$$

with mass dimensions $[\Phi] = 1$, $[\mathcal{M}_4] = 2$ and $[J] = 3$.

In order to relate (3.13) to (3.8), we have to convert the latter into a first-order theory as well. This is readily done by expressing the velocities $\partial_t\phi_1$ and $\partial_t\phi_2$ in terms of the momenta

$$\begin{aligned}\tilde\pi_1 &\equiv \frac{\partial\mathcal{L}^2}{\partial(\partial_t\phi_1)} = -\partial_t\phi_1\\ \tilde\pi_2 &\equiv \frac{\partial\mathcal{L}^2}{\partial(\partial_t\phi_2)} = \partial_t\phi_2\end{aligned} \tag{3.15}$$

so that

$$\mathcal{H}^2[\phi_1, \phi_2, \tilde\pi_1, \tilde\pi_2] = -\tfrac{1}{2}\tilde\pi_1^2 + \tfrac{1}{2}\tilde\pi_2^2 - \tfrac{1}{2}\phi_1 M_1^2\phi_1 + \tfrac{1}{2}\phi_2 M_2^2\phi_2 + j(\phi_1 + \phi_2). \tag{3.16}$$

The Helmholtz Lagrangian that yields the canonical equations is now

$$\mathcal{L}_H^2 = \tfrac{1}{2}\Theta^T \mu\Sigma\partial_t\Theta + \tfrac{1}{2}\Theta^T \mathcal{M}_2\Theta - J^T \mathcal{Z}\Theta \tag{3.17}$$

where

$$\Theta \equiv \begin{pmatrix} \mu^{-1}\tilde{\pi}_1 \\ \phi_1 \\ \mu^{-1}\tilde{\pi}_2 \\ \phi_2 \end{pmatrix} \qquad \mathcal{M}_2 \equiv \begin{pmatrix} \mu^2 & 0 & 0 & 0 \\ 0 & M_1^2 & 0 & 0 \\ 0 & 0 & -\mu^2 & 0 \\ 0 & 0 & 0 & -M_2^2 \end{pmatrix} \tag{3.18}$$

with mass dimensions $[\Theta] = 1$ and $[\mathcal{M}_2] = 2$, and $\mathcal{Z}$ is any matrix with the fourth row equal to $(0, 1, 0, 1)$.

The field redefinition analogous to the diagonalizing equations (3.7) is now a $4 \times 4$ mixing of fields given by

$$\Phi = \mathcal{X}\Theta \tag{3.19}$$

where the invertible matrix

$$\mathcal{X} \equiv \begin{pmatrix} 0 & -\frac{M_1^2}{\langle 12\rangle} & 0 & -\frac{M_2^2}{\langle 12\rangle} \\ -1 & 0 & 1 & 0 \\ -\frac{M_2^2}{\langle 12\rangle} & 0 & \frac{M_1^2}{\langle 12\rangle} & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \tag{3.20}$$

verifies

$$\mathcal{X}^T \Sigma \mathcal{X} = \Sigma \tag{3.21}$$

$$\mathcal{X}^T \mathcal{M}_4 \mathcal{X} = \mathcal{M}_2 \tag{3.22}$$

so we can identify $\mathcal{Z} = \mathcal{X}$.

We thus see that (3.19) translates (3.13) into (3.17), and therefore the Lagrangians (3.9) and (3.8) are again seen to be equivalent. The derivation of the matrix $\mathcal{X}$ is cumbersome but contains interesting details that justify the appendix. Notice that the components of $\Phi$ are expressed by (3.19) in terms of the components of $\Theta$ *and* of their space derivatives. This is not surprising as long as $\pi_1$, given by (3.11), contains space derivatives of $\phi$ as well.

Though the plain non-covariant Ostrogradski method we have just implemented eventually shows up the Lorentz invariance, the readiness of the explicitly covariant procedure formerly introduced in this section is apparent. The non-covariant approach using the canonical Hamiltonian and mechanical momenta is rigorous and validates the former, but involves more bulky diagonalizing matrices with elements that contain space derivatives.

## 4. $N = 4$ and higher even $N$ theories

We treat the $N = 4$ theory with the far more practical Lorentz-invariant method of the previous section. Otherwise one would have to face the diagonalization of $8 \times 8$ matrices analogous to $\hat{\mathcal{M}}_2$ and $\hat{\mathcal{M}}_4$ in the appendix. Our Lagrangian is now

$$\mathcal{L}^8 = -\frac{1}{2}\frac{\mu^6}{M}\phi[\![1]\!][\![2]\!][\![3]\!][\![4]\!]\phi - j\phi \tag{4.1}$$

where the mass dimensions $[\mu] = [\phi] = 1, [M] = 12$ and $[j] = 3$ are such that $[\mathcal{L}^8] = 4$. Taking $M = \langle 12\rangle\langle 13\rangle\langle 14\rangle\langle 23\rangle\langle 24\rangle\langle 34\rangle$, equation (4.1) treats the masses $m_i$ $(i = 1, \ldots, 4)$ on an equal footing, which is apparent in the propagator

$$-\frac{\mu^{-6}M}{[\![1]\!][\![2]\!][\![3]\!][\![4]\!]} = \frac{\langle 1\rangle}{[\![1]\!]} - \frac{\langle 2\rangle}{[\![2]\!]} + \frac{\langle 3\rangle}{[\![3]\!]} - \frac{\langle 4\rangle}{[\![4]\!]} \tag{4.2}$$

where $\langle i\rangle \equiv \mu^{-6} M \prod_{j\neq i} \frac{1}{\langle ij\rangle}$ (remind the ordering convention $i < j$) with mass dimensions $[\langle i\rangle] = 0$.

As for (3.2), the propagator expansion (4.2) suggests that the lower-derivative equivalent theory should now be

$$\mathcal{L}^2 = \frac{1}{2}\frac{1}{\langle 1\rangle}\phi_1[\![1]\!]\phi_1 - \frac{1}{2}\frac{1}{\langle 2\rangle}\phi_2[\![2]\!]\phi_2 + \frac{1}{2}\frac{1}{\langle 3\rangle}\phi_3[\![3]\!]\phi_3 - \frac{1}{2}\frac{1}{\langle 4\rangle}\phi_4[\![4]\!]\phi_4 - j(\phi_1+\phi_2+\phi_3+\phi_4). \tag{4.3}$$

We derive this Lagrangian from (4.1) in the following. In matrix form, (4.3) reads

$$\mathcal{L}^2 = \tfrac{1}{2}\tau^T[\![1]\!]I\tau + \tfrac{1}{2}\tau^T\mathcal{M}_2\tau - J^T F\tau \tag{4.4}$$

where

$$\tau \equiv \begin{pmatrix} \langle 1\rangle^{-\frac{1}{2}}\phi_1 \\ -\mathrm{i}\langle 2\rangle^{-\frac{1}{2}}\phi_2 \\ \langle 3\rangle^{-\frac{1}{2}}\phi_3 \\ -\mathrm{i}\langle 4\rangle^{-\frac{1}{2}}\phi_4 \end{pmatrix} \qquad J \equiv \begin{pmatrix} 0\\0\\0\\j \end{pmatrix} \qquad \mathcal{M}_2 \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\langle 12\rangle & 0 & 0 \\ 0 & 0 & -\langle 13\rangle & 0 \\ 0 & 0 & 0 & -\langle 14\rangle \end{pmatrix} \tag{4.5}$$

$I$ is the $4\times 4$ identity, and $F$ is any matrix with the fourth row equal to $(\langle 1\rangle^{\frac{1}{2}}, \mathrm{i}\langle 2\rangle^{\frac{1}{2}}, \langle 3\rangle^{\frac{1}{2}}, \mathrm{i}\langle 4\rangle^{\frac{1}{2}})$.

By dropping total derivatives we express (4.1) in a standard form involving derivatives of the lowest possible order, namely

$$\mathcal{L}^8[\phi, [\![1]\!]\phi, [\![1]\!]^2\phi] = -\frac{1}{2}\frac{\mu^6}{M}\{([\![1]\!]^2\phi)^2 - S([\![1]\!]\phi)([\![1]\!]^2\phi) + p([\![1]\!]\phi)^2 - P\phi([\![1]\!]\phi)\} - j\phi \tag{4.6}$$

where $S \equiv \langle 12\rangle + \langle 13\rangle + \langle 14\rangle$, $p \equiv \langle 12\rangle\langle 13\rangle + \langle 12\rangle\langle 14\rangle + \langle 13\rangle\langle 14\rangle$, and $P \equiv \langle 12\rangle\langle 13\rangle\langle 14\rangle$.

Ostrogradski-like momenta are defined as follows

$$\begin{aligned} \pi_2 &= \frac{\partial\mathcal{L}^8}{\partial([\![1]\!]^2\phi)} = -\frac{\mu^6}{M}([\![1]\!]^2\phi) + \frac{\mu^6 S}{2M}[\![1]\!]\phi \\ \pi_1 &= \frac{\partial\mathcal{L}^8}{\partial([\![1]\!]\phi)} + [\![1]\!]\pi_2. \end{aligned} \tag{4.7}$$

From the first of (4.7) the highest derivative is worked out, namely

$$[\![\mathbf{1}]\!]^{\mathbf{2}}\phi[\pi_2, [\![1]\!]\phi] = -\frac{M}{\mu^6}\pi_2 + \frac{S}{2}([\![1]\!]\phi). \tag{4.8}$$

The 'Hamiltonian' functional is

$$\mathcal{H}^8[\psi_1, \psi_2, \pi_1, \pi_2] = \pi_2[\![\mathbf{1}]\!]^{\mathbf{2}}\phi + \pi_1\psi_2 - \mathcal{L}^8[\psi_1, \psi_2, [\![\mathbf{1}]\!]^{\mathbf{2}}\phi] \tag{4.9}$$

where $\psi_1 \equiv \phi$ and $\psi_2 \equiv [\![1]\!]\phi$. Its canonical equations can be derived from the Lagrangian

$$\mathcal{L}^8_H = \tfrac{1}{2}\Phi^T[\![1]\!]\mathcal{K}\Phi + \tfrac{1}{2}\Phi^T\mathcal{M}_8\Phi - J^T\Phi \tag{4.10}$$

where $J$ is the same as in (4.5),

$$\Phi \equiv \begin{pmatrix} \mu^2\pi_2 \\ \mu^{-2}\psi_2 \\ \pi_1 \\ \psi_1 \end{pmatrix} \qquad \mathcal{K} \equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$
$$\mathcal{M}_8 \equiv \begin{pmatrix} \mu^{-10}M & -\frac{S}{2} & 0 & 0 \\ -\frac{S}{2} & -\frac{\mu^{-10}}{M}\left(p - \frac{S^2}{4}\right) & -\mu^2 & \frac{\mu^2}{2\langle 1\rangle} \\ 0 & -\mu^2 & 0 & 0 \\ 0 & \frac{\mu^2}{2\langle 1\rangle} & 0 & 0 \end{pmatrix}. \tag{4.11}$$

*Prior* to its diagonalization we write (4.10) in the form

$$\mathcal{L}_H^8 = \tfrac{1}{2}\Omega^T [\![1]\!] I\Omega + \tfrac{1}{2}\Omega^T \hat{\mathcal{M}}_8 \Omega - J^T \mathcal{D}^T \Omega \tag{4.12}$$

where $\Omega \equiv (\mathcal{D}^T)^{-1}\Phi$, with

$$\mathcal{D} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ -\mathrm{i} & \mathrm{i} & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -\mathrm{i} & \mathrm{i} \end{pmatrix} \tag{4.13}$$

and

$$\hat{\mathcal{M}}_8 \equiv \mathcal{D}\mathcal{M}_8\mathcal{D}^{-1} = \frac{1}{2} \begin{pmatrix} M_- - S & -\mathrm{i}M_+ & -\mu^2 1_- & \mathrm{i}\mu^2 1_+ \\ -\mathrm{i}M_+ & -(M_- + S) & -\mathrm{i}\mu^2 1_- & -\mu^2 1_+ \\ -\mu^2 1_- & -\mathrm{i}\mu^2 1_- & 0 & 0 \\ \mathrm{i}\mu^2 1_+ & -\mu^2 1_+ & 0 & 0 \end{pmatrix} \tag{4.14}$$

with $M_\pm \equiv \frac{M}{\mu^{10}} \pm \frac{\mu^{10}}{M}(p - \frac{S^2}{4})$ and $1_\pm \equiv 1 \pm \frac{1}{2\langle 1\rangle}$.

Now the task is to establish the equivalence of (4.12) and (4.4). One may first check that the eigenvalues $\lambda_i$ $(i = 1, \ldots, 4)$ of $\hat{\mathcal{M}}_8$ are the diagonal elements of $\mathcal{M}_2$ in (4.5). The orthogonal matrix $T$ that diagonalizes $\hat{\mathcal{M}}_8$ is obtained by working out its orthonormal eigenvectors $|\lambda_i\rangle$ with the suitable sign, and arranging them in columns. These are

$$|\lambda_1\rangle = \frac{\langle 1\rangle^{\frac{1}{2}}}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1_+ \\ -\mathrm{i}1_- \end{pmatrix}$$

$$|\lambda_j\rangle = \frac{\mathrm{i}^{(1-\delta_{3j})}\langle j\rangle^{\frac{1}{2}}}{\sqrt{2}[-\frac{2}{\mu^{10}}M + 2\langle 1j\rangle - S]} \begin{pmatrix} \frac{2}{\mu^2}[-\frac{\mu^4}{\langle 1\rangle} + \langle 1j\rangle(2\langle 1j\rangle - S - M_-)] \\ \mathrm{i}\frac{2}{\mu^2}[-\frac{\mu^4}{\langle 1\rangle} + \langle 1j\rangle M_+] \\ 1_-[-2\mu^{-10}M + 2\langle 1j\rangle - S] \\ -\mathrm{i}1_+[-2\mu^{-10}M + 2\langle 1j\rangle - S] \end{pmatrix} \tag{4.15}$$

where $j = 2, 3, 4$. If $I$ is the identity matrix, we therefore have

$$T^T I T = I \qquad T^T \hat{\mathcal{M}}_8 T = \mathcal{M}_2 \tag{4.16}$$

and the fourth row of $\mathcal{D}^T T$ can be seen to be $(\langle 1\rangle^{\frac{1}{2}}, \mathrm{i}\langle 2\rangle^{\frac{1}{2}}, \langle 3\rangle^{\frac{1}{2}}, \mathrm{i}\langle 4\rangle^{\frac{1}{2}})$, i.e. it has the required form for $F$. Then, by taking $\Omega = T\tau$, (4.12) is identical to (4.4).

The general case for even $N \geqslant 6$ in the covariant treatment would involve $\frac{N}{2}$ Ostrogradski-like momenta and the diagonalization of an $N \times N$ mass matrix. The non-covariant Ostrogradski method introduced in section 3, which reduces the theory to a first differential-order form, would now involve $2N \times 2N$ matrices. In both treatments the procedure would follow analogous paths, albeit with the occurrence of intractable eigenvector and diagonalization problems.

## 5. $N = 3$ and higher odd $N$ theories

For $N = 3$, the HD Lagrangian

$$\mathcal{L}^6 = -\frac{1}{2}\frac{\mu^2}{M}\phi[\![1]\!][\![2]\!][\![3]\!]\phi - j\phi \tag{5.1}$$

where $M \equiv \langle 12\rangle\langle 13\rangle\langle 23\rangle$ and $[\mathcal{L}^6] = 4$, yields the propagator

$$-\frac{\mu^{-2}M}{[\![1]\!][\![2]\!][\![3]\!]} = -\frac{\mu^{-2}\langle 23\rangle}{[\![1]\!]} + \frac{\mu^{-2}\langle 13\rangle}{[\![2]\!]} - \frac{\mu^{-2}\langle 12\rangle}{[\![3]\!]}. \tag{5.2}$$

Then, the expected equivalent second-order theory is

$$\mathcal{L}^2 = -\frac{1}{2}\frac{\mu^2}{\langle 23\rangle}\phi_1[\![1]\!]\phi_1 + \frac{1}{2}\frac{\mu^2}{\langle 13\rangle}\phi_2[\![2]\!]\phi_2 - \frac{1}{2}\frac{\mu^2}{\langle 12\rangle}\phi_3[\![3]\!]\phi_3 - j(\phi_1+\phi_2+\phi_3). \tag{5.3}$$

Already for $N = 3$, the non-covariant Ostrogradski method becomes exceedingly cumbersome. In fact, it reduces both (5.1) and (5.3) to first differential order in time. Proving the equivalence of those theories then involves the diagonalization of $6\times 6$ matrices (the counterpart of $\hat{\mathcal{M}}_4$ and $\hat{\mathcal{M}}_2$ in (A.4)), although with a reasonable amount of work it can still be checked that both mass matrices have the same eigenvalues, namely $\pm\mu M_1$, $\pm\mu M_2$ and $\pm\mu M_3$. Finding the eigenvectors and building up the compound diagonalizing transformation does not justify the effort.

For the odd $N$ theories, the covariant method exhibits an interesting feature. Without loss of generality we again single out the Klein–Gordon operator $[\![1]\!]$ and write (5.1) as

$$\mathcal{L}^6[\phi,[\![1]\!]\phi,[\![1]\!]^2\phi] = -\frac{1}{2}\frac{\mu^2}{M}\{(\rrbracket 1\rrbracket\phi)(\rrbracket 1\rrbracket^2\phi) - S(\rrbracket 1\rrbracket\phi)^2 + P\phi(\rrbracket 1\rrbracket\phi)\} - j\phi \tag{5.4}$$

where now $S \equiv \langle 12\rangle + \langle 13\rangle$ and $P \equiv \langle 12\rangle\langle 13\rangle$.

The momenta are

$$\begin{aligned}\pi_2 &= \frac{\partial\mathcal{L}^6}{\partial([\![1]\!]^2\phi)} = -\frac{1}{2}\frac{\mu^2}{M}[\![1]\!]\phi \\ \pi_1 &= \frac{\partial\mathcal{L}^6}{\partial([\![1]\!]\phi)} + [\![1]\!]\pi_2 = -\frac{\mu^2}{M}[\![1]\!]^2\phi + \frac{\mu^2}{M}S[\![1]\!]\phi - \frac{1}{2}\frac{\mu^2}{M}P\phi.\end{aligned} \tag{5.5}$$

Unlike in (4.7), the highest derivative is now worked out of $\pi_1$ (instead of $\pi_2$), namely

$$[\![1]\!]^2\phi[\phi,[\![1]\!]\phi,\pi_1] = -\frac{M}{\mu^2}\pi_1 + S[\![1]\!]\phi - \frac{1}{2}P\phi \tag{5.6}$$

and, in terms of the coordinates $\pi_1, \pi_2, \psi_1 \equiv \phi$ and $\psi_2 \equiv [\![1]\!]\phi$, the 'Hamiltonian' reads

$$\mathcal{H}^6[\psi_1,\psi_2,\pi_1,\pi_2] = \pi_2[\![1]\!]^2\phi + \pi_1\psi_2 - \mathcal{L}^6[\psi_1,\psi_2,[\![1]\!]^2\phi]. \tag{5.7}$$

The Helmholtz Lagrangian is

$$\begin{aligned}\mathcal{L}^6_H[\psi_1,\psi_2,\pi_1,\pi_2] &= \pi_2[\![1]\!]\psi_2 + \pi_1[\![1]\!]\psi_1 + \frac{M}{\mu^2}\pi_1\pi_2 - S\pi_2\psi_2 \\ &\quad + \frac{1}{2}P\pi_2\psi_1 - \frac{1}{2}\pi_1\psi_2 - \frac{\mu^2}{4M}P\psi_1\psi_2 - j\psi_1.\end{aligned} \tag{5.8}$$

The distinctive feature of the odd $N$ cases is that the first of (5.5), namely $\pi_2 = -\frac{1}{2}\frac{\mu^2}{M}\psi_2$, is a constraint that guarantees the relationship $[\![1]\!]\psi_1 = \psi_2$, so one just has $N$ degrees of freedom. For even $N$ it arises directly as an equation of motion. Moreover, unlike the Dirac Lagrangian for spin-$\frac{1}{2}$ fields or the constraints introduced by means of multipliers, the constraint above can be freely imposed on the Lagrangian since it does not eliminate the dependence on the remaining variables $\psi_1$ and $\pi_1$. Thus, (5.8) can be expressed in terms of only the three fields $\psi_1$, $\pi_1$ and $\pi_2$:

$$\mathcal{L}^6_H[\psi_1,\pi_1,\pi_2] = \tfrac{1}{2}\Phi^T[\![1]\!]\mathcal{K}'\Phi + \tfrac{1}{2}\Phi^T\mathcal{M}_3\Phi - J^T\Phi \tag{5.9}$$

where

$$\Phi \equiv \begin{pmatrix} \mu^2\pi_2 \\ \pi_1 \\ \phi \end{pmatrix} \qquad J \equiv \begin{pmatrix} 0 \\ 0 \\ j \end{pmatrix}$$
$$\mathcal{K}' \equiv \begin{pmatrix} -4\frac{M}{\mu^6} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{5.10}$$
$$\mathcal{M}_3 \equiv \begin{pmatrix} 4\frac{MS}{\mu^6} & 2\frac{M}{\mu^4} & \frac{P}{\mu^2} \\ 2\frac{M}{\mu^4} & 0 & 0 \\ \frac{P}{\mu^2} & 0 & 0 \end{pmatrix}.$$

The Lagrangian (5.9) is expected to be equivalent to (5.3), which in matrix form reads

$$\mathcal{L}^2 = -\tfrac{1}{2}\tau^T[\![1]\!]I\tau + \tfrac{1}{2}\tau^T\mathcal{M}_2'\tau - J^TG\tau \tag{5.11}$$

where $I$ is the $3\times 3$ identity matrix,

$$\tau \equiv \begin{pmatrix} \mu\langle 23\rangle^{-\frac{1}{2}}\phi_1 \\ \mathrm{i}\mu\langle 13\rangle^{-\frac{1}{2}}\phi_2 \\ \mu\langle 12\rangle^{-\frac{1}{2}}\phi_3 \end{pmatrix} \qquad \mathcal{M}_2' \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & \langle 12\rangle & 0 \\ 0 & 0 & \langle 13\rangle \end{pmatrix} \tag{5.12}$$

and $G$ is any matrix with the third row given by $(\mu^{-1}\langle 23\rangle^{\frac{1}{2}}, -\mathrm{i}\mu^{-1}\langle 13\rangle^{\frac{1}{2}}, \mu^{-1}\langle 12\rangle^{\frac{1}{2}})$.

The transformation of (5.9) into (5.11) is performed by the field redefinition

$$\Phi = \mathcal{D}'T\tau \tag{5.13}$$

where

$$\mathcal{D}' \equiv \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{\mu^3}{\sqrt{2M}} & 0 & 0 \\ 0 & -\mathrm{i} & -1 \\ 0 & -\mathrm{i} & 1 \end{pmatrix} \tag{5.14}$$

and $T$ is an orthogonal matrix built up with the eigenvectors of $\mathcal{D}'^T\mathcal{M}_3\mathcal{D}'$, namely

$$T = \frac{\mu}{2\sqrt{2}}\langle 23\rangle^{-\frac{1}{2}}\begin{pmatrix} 0 & \mathrm{i}\frac{2\sqrt{2}}{\mu}\langle 12\rangle^{\frac{1}{2}} & -\frac{2\sqrt{2}}{\mu}\langle 13\rangle^{\frac{1}{2}} \\ -\mathrm{i}\frac{P_-}{P} & \frac{P_+}{\sqrt{M}}\langle 12\rangle^{-\frac{1}{2}} & \mathrm{i}\frac{P_+}{\sqrt{M}}\langle 13\rangle^{-\frac{1}{2}} \\ \frac{P_+}{P} & \mathrm{i}\frac{P_-}{\sqrt{M}}\langle 12\rangle^{-\frac{1}{2}} & -\frac{P_-}{\sqrt{M}}\langle 13\rangle^{-\frac{1}{2}} \end{pmatrix} \tag{5.15}$$

with $P_\pm \equiv P \pm \mu^{-2}2M$.

Then $\mathcal{D}'^T\mathcal{K}'\mathcal{D}' = -I$ and $T^T\mathcal{D}'^T\mathcal{M}_3\mathcal{D}'T = \mathcal{M}_2'$. One may also check that $\mathcal{D}'T$ has the same third row required for $G$.

The covariant treatment of the general odd $N \geqslant 5$ case proceeds along the same lines. Initially $(N+1)/2$ Ostrogradski coordinates plus the corresponding momenta occur. Again the definition of the highest momentum yields a constraint with the same meaning as above, while the highest field derivative is worked out of the next momentum definition. Then one faces the diagonalization of a Helmholtz Lagrangian depending on just $N$ fields.

Already in the $N = 3$ case one might have chosen not to implement the constraint on the Lagrangian (5.8) and allowed it to arise in the equations of motion. These equations are the canonical ones for the Hamiltonian (5.7) and involve an even number of variables, as required by phase space. Thus one keeps the dependence of the Lagrangian (5.8) on the four fields $\psi_1, \psi_2, \pi_1$ and $\pi_2$. Notwithstanding this enlarged dependence, it may still be diagonalized by new fields $\phi_1, \phi_2, \phi_3$ and $\zeta$, the (expected) surprise being that $\zeta$ does not couple to the source $j$. It is a spurious field, which moreover vanishes when the constraint is implemented. We skip here the details of this derivation.

## 6. Conclusions

We have shown the physical equivalence between relativistic HD theories of a scalar field and a reduced second differential-order counterpart. The free part of the HD scalar theories can always be brought to the form (2.19) by integrating by parts, and the only limitation of our procedure is the non-degeneracy of the resulting Klein–Gordon masses, i.e. we consider regular Lagrangians. The existence of a lower-derivative version is already suggested by the algebraic decomposition of the HD propagator into a sum of second-order pieces showing (physical and ghost) particle poles. The order-reducing programme we have developed relies on an extension of the Legendre transformation procedure, on the use of the modified action principle (Helmholtz Lagrangian) and on a suitable diagonalization. A basic ingredient of this programme is the Ostrogradski formalism, which we have extended to field systems.

Two approaches have been worked out. The first one follows Ostrogradski more closely by defining generalized momenta and Hamiltonians with a standard mechanical meaning, at the price of treating time separately and losing the explicit Lorentz invariance. It validates a second and more powerful one which is explicitly Lorentz invariant. The rigorous non-invariant phase-space analysis also strongly backs the formal covariant methods used in HD gravity, where $R_{\mu\nu}[g, \partial g, \partial^2 g]$ and $\Box h_{\mu\nu}$ (in the linearized theory) are used in the Legendre transformation.

The HD theories of the scalar field we have considered are generalized Klein–Gordon theories, and hence of $2N$ differential order according to the number $N$ of Klein–Gordon operators involved. While the non-invarint approach treats all the theories with equal footing, the odd $N$ and the even $N$ cases feature qualitative differences in the invariant method. Also the ratio of physical versus ghost fields varies. For even $N$ one finds $N/2$ fields of each type. For odd $N$ one has $(N-1)/2$ ghost (physical) and $(N+1)/2$ physical (ghost) fields according to the overall negative (positive) sign of the free part of the HD Lagrangian. The squared masses may be shifted by an arbitrary common ammount, since only their differences are involved in the procedure. Then any of them may be zero (only one in this case), or tachyonic.

On the other hand, the non-invariant procedure gets exceedingly cumbersome already for $N = 3$, in contrast with the (more compact) invariant one which remains tractable up to $N = 4$ at least. Both approaches are applicable to higher $N$, only at the price of increasing the length of the calculations (namely analitically diagonalizing $N \times N$ matrices). An intriguing feature of the odd $N$ cases when treated with the invariant method is the occurrence of a constraint on an otherwise overabundant set of Ostrogradski-like coordinates and momenta, together with a less conventional way of working out the highest field derivative. Ignoring the constraint causes the appearance of a spurious decoupled scalar field.

## Acknowledgment

We are indebted to Dr J León for the careful reading of the manuscript and useful suggestions.

## Appendix

The problem of finding a matrix $\mathcal{X}$ with the properties (3.21) and (3.22) can be brought to the one of diagonalizing a symmetric $4 \times 4$ matrix with pure real and imaginary elements.

The procedure is somehow tricky since there is no similarity-like transformation that brings the symplectic matrix $\Sigma$ to the identity matrix, thus preventing a plain use of the weaponry of orthonormal transformations. We introduce the diagonal matrices $f \equiv \mathrm{diag}(\mathrm{i}, 1, 1, -\mathrm{i})$ and $g \equiv \mathrm{diag}(1, \mathrm{i}, \mathrm{i}, -1)$ so that

$$\Sigma = gKf \qquad \text{where } K \equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{A.1}$$

Taking $f \neq g$ does not compromise the uniqueness of the transformation $\Phi \to \Theta$ as shown at the end.

Now we transform the symmetric matrix $K$ into the $4 \times 4$ identity by a similarity transformation

$$\mathcal{D} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ -\mathrm{i} & \mathrm{i} & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -\mathrm{i} & \mathrm{i} \end{pmatrix} \tag{A.2}$$

so that

$$\mathcal{D}\mathcal{K}\mathcal{D}^T = \mathcal{D}g^{-1}\Sigma f^{-1}\mathcal{D}^T = I. \tag{A.3}$$

This same transformation converts $\mathcal{M}_4$ and $\mathcal{M}_2$ into

$$\begin{aligned} \hat{\mathcal{M}}_4 &= \mathcal{D}g^{-1}\mathcal{M}_4 f^{-1}\mathcal{D}^T \\ \hat{\mathcal{M}}_2 &= \mathcal{D}g^{-1}\mathcal{M}_2 f^{-1}\mathcal{D}^T. \end{aligned} \tag{A.4}$$

Notice that $\hat{\mathcal{M}}_2$ and $\hat{\mathcal{M}}_4$ are symmetric as well. This is a consequence of the vanishing of some critical elements in both matrices. One then verifies that they have the same eigenvalues, namely $-\mathrm{i}\mu M_1, \mathrm{i}\mu M_1, \mathrm{i}\mu M_2$ and $-\mathrm{i}\mu M_2$, so that there exist orthogonal matrices $R$ and $T$ such that

$$T^T \hat{\mathcal{M}}_4 T = R^T \hat{\mathcal{M}}_2 R = \mathrm{i}\mu \,\mathrm{diag}(-M_1, M_1, M_2, -M_2) \tag{A.5}$$

while conserving the euclidean metric $I$:

$$R^T I R = T^T I T = I. \tag{A.6}$$

With the orthonormal eigenvectors as columns one obtains

$$R = \frac{1}{2\sqrt{\mu}} \begin{pmatrix} -R_1^+ & -\mathrm{i}R_1^- & 0 & 0 \\ -\mathrm{i}R_1^- & R_1^+ & 0 & 0 \\ 0 & 0 & R_2^+ & \mathrm{i}R_2^- \\ 0 & 0 & \mathrm{i}R_2^- & -R_2^+ \end{pmatrix} \tag{A.7}$$

where

$$R_i^{\pm} \equiv \frac{M_i \pm \mu}{\sqrt{M_i}} \tag{A.8}$$

and

$$T = \frac{1}{2\langle 12\rangle\sqrt{\mu}} \begin{pmatrix} T_1^+ & -\mathrm{i}T_1^- & -T_2^- & \mathrm{i}T_2^+ \\ \mathrm{i}T_1^- & T_1^+ & -\mathrm{i}T_2^+ & -T_2^- \\ P_1^- & \mathrm{i}P_1^+ & P_2^+ & \mathrm{i}P_2^- \\ \mathrm{i}P_1^+ & -P_1^- & \mathrm{i}P_2^- & -P_2^+ \end{pmatrix} \tag{A.9}$$

where

$$\begin{aligned} T_i^{\pm} &\equiv \sqrt{M_i}\left(\mu\sqrt{M_i} \pm \langle 12\rangle\right) \\ P_i^{\pm} &\equiv \frac{\langle 12\rangle\sqrt{M_i}}{M_i^2}\left(\frac{P}{\langle 12\rangle} \pm \mu M_i\right). \end{aligned} \tag{A.10}$$

Notice that one has pure real and imaginary matrix elements and vector components, and that the norm of a vector, defined as $|V| \equiv V^T V$, may be imaginary as well. Since $M_i^2 \equiv m_i^2 - \triangle$, a regularization (the dimensional one, for instance) is understood such that $R$ and $T$ have well-defined elements.

Finally, from (A.4) and (A.5) one finds

$$Y\mathcal{M}_4 W = \mathcal{M}_2 \tag{A.11}$$

where $W \equiv f^{-1}\mathcal{D}^T T R^T \mathcal{D}^{-1^T} f$ and $Y \equiv g\mathcal{D}^{-1} R T^T \mathcal{D} g^{-1}$. The matrix $W$ has some imaginary elements and the fourth row is not (0101), so that it is not suitable to relate the real vectors $\Phi$ and $\Theta$ as in (3.19) yet. Moreover, $Y \neq W^T$. However, one may check that

$$\begin{pmatrix} \mathrm{i} & 0 & 0 & 0 \\ 0 & \mathrm{i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} Y = \mathcal{X}^T \qquad \text{where } \mathcal{X} \equiv W \begin{pmatrix} -\mathrm{i} & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{A.12}$$

is the matrix given in (3.20), so that (A.11) writes

$$\mathcal{X}^T \mathcal{M}_4 \mathcal{X} = \mathcal{M}_2. \tag{A.13}$$

Furthermore, from (A.3) and (A.6) one has that

$$\mathcal{X}^T \Sigma \mathcal{X} = \Sigma. \tag{A.14}$$

The fourth row of $\mathcal{X}$ has the desired elements (0101) only if suitable signs are chosen for the eigenvectors that build up $R$ and $T$, so that the handedness of the frame is conserved by $\mathcal{X}$. We stress that $\mathcal{X}$ is also well defined as a differential operator, and that the regularization is needed only for defining the intermediate operators $T$ and $R^T$. At the end of the process the regularization can be put off.

## References


[1] Podolski B and Schwed P 1948 *Rev. Mod. Phys.* **20** 40

[2] Stelle K S 1978 *Gen. Rel. Grav.* **9** 353
Bartoli A and Julve J 1994 *Nucl. Phys.* B **425** 277

[3] Barci D G, Bollini C G and Rocca M C 1995 *Int. J. Mod. Phys.* A **10** 1737

[4] Pimentel B M and Teixeira R G 1998 *Nuovo Cimento* B **113** 805

[5] Gross D J and Witten E 1986 *Nucl. Phys.* B **277** 1
Metsaev R R and Tseytlin A A 1987 *Phys. Lett.* **185B** 52
Bento M C and Bertolami O 1989 *Phys. Lett.* B **228** 348

[6] Birrell N D and Davies P C W 1982 *Quantum Fields in Curved Space* (Cambridge: Cambridge University Press)

[7] Stelle K S 1977 *Phys. Rev.* D **16** 953

[8] Buchbinder I L, Odintsov S D and Shapiro I L 1992 *Effective Action in Quantum Gravity* (Bristol: IOP Publishing)

[9] Goldman T, Pérez-Mercader J, Cooper F and Nieto M M 1992 *Phys. Lett.* **281** 219
Elizalde E, Odintsov S D and Romeo A 1995 *Phys. Rev.* D **51** 1680

[10] Ferraris M and Kijowski J 1982 *Gen. Rel. Grav.* **14** 165
Jakubiec A and Kijowski J 1988 *Phys. Rev.* D **37** 1406
Magnano G, Ferraris M and Francaviglia M 1987 *Gen. Rel. Grav.* **19** 465

Magnano G, Ferraris M and Francaviglia M 1990 *J. Math. Phys.* **31** 378
Magnano G, Ferraris M and Francaviglia M 1990 *Class. Quantum Grav.* **7** 557
[11] Alonso J C, Barbero F, Julve J and Tiemblo A 1994 *Class. Quantum Grav.* **11** 865
[12] Alonso J C and Julve J 1993 Particle contents of higher order gravity *Classical and Quantum Gravity (Proc. 1st Iberian Meeting on Gravity, Evora, Portugal, 1992)* (Singapore: World Scientific) p 301
[13] Barth N H and Christensen S M 1983 *Phys. Rev.* D **28** 1876
[14] Jansen K, Kuti J and Liu Ch 1993 *Phys. Lett.* B **309** 119
Jansen K, Kuti J and Liu Ch 1993 *Phys. Lett.* B **309** 127
[15] Whittaker E T 1904 *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies* (Cambridge: Cambridge University Press) Pioneering work by M Ostrogradski and W F Donkin is quoted here
[16] Saunders D J and Crampin M 1990 *J. Phys. A: Math. Gen.* **23** 3169
Gotay M J 1991 *Mechanics, Analysis and Geometry: 200 Years after Lagrange* ed M Francaviglia (Amsterdam: Elsevier)